Brownian Motion
Part I - The Scaled Random Walk

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A scaled random walk is a trajectory that consists of taking consecutive random steps in discrete time. At each time step the random walk can either increase or decrease in value by a fixed amount. If each step is independent of past steps then the random walk is a Markov process. As the step size in the random walk tends to zero (and the number of steps increases proportionally) the random walk converges to a Wiener process, which is also known as Brownian motion. We will use Brownian motion to model the future value (and path taken) of risky assets.

We will define M_i to be the value of a scaled random walk after the ith time step. We want to find the path followed during the time interval [0,T] where there are N steps in the path such that the length of each time step is $\Delta_t = T/N$. The path is a scaled random walk that starts out at zero and at the end of each time step can either increase by $\sigma\sqrt{\Delta_t}$ with probability p or decrease by $\sigma\sqrt{\Delta_t}$ with probability $1-p$. If we define X_i to be the change in the random walk after the ith time step then...

$$X_i = M_i - M_{i-1} \text{ ...where... } X_i = \sigma\sqrt{\Delta_t} \text{ ...or... } X_i = -\sigma\sqrt{\Delta_t} \tag{1}$$

The value of the random walk at time T is the value of the random walk at time zero plus the sum of the changes in the random walk during the interval [0,T]. In equation form the value of the random walk at time T is...

$$M_T = M_0 + \sum_{i=1}^{N} X_i \tag{2}$$

The expected value of the change in the random walk at any given time step i is...

$$\begin{aligned}\mathbb{E}\Big[X_i\Big] &= \Big[p \times \sigma\sqrt{\Delta_t}\Big] + \Big[(1-p) \times -\sigma\sqrt{\Delta_t}\Big] \\ &= p\,\sigma\sqrt{\Delta_t} - \sigma\sqrt{\Delta_t} + p\,\sigma\sqrt{\Delta_t} \\ &= \sigma\sqrt{\Delta_t}\,(2p-1)\end{aligned} \tag{3}$$

The expected value of the square of the change in the random walk at any given time step i is...

$$\begin{aligned}\mathbb{E}\Big[X_i^2\Big] &= \Big[p \times \Big\{\sigma\sqrt{\Delta_t}\Big\}^2\Big] + \Big[(1-p) \times \Big\{-\sigma\sqrt{\Delta_t}\Big\}^2\Big] \\ &= p\,\sigma^2\Delta_t + (1-p)\,\sigma^2\Delta_t \\ &= p\,\sigma^2\Delta_t + \sigma^2\Delta_t - p\,\sigma^2\Delta_t \\ &= \sigma^2\Delta_t\end{aligned} \tag{4}$$

The mean of each increment in the random walk using Equation (3) above is...

$$\begin{aligned}mean &= \mathbb{E}\Big[X_i\Big] \\ &= \sigma\sqrt{\Delta_t}\,(2p-1)\end{aligned} \tag{5}$$

The variance of each increment in the random walk using Equation (3) and Equation (4) above is...

$$\begin{aligned}
variance &= \mathbb{E}\Bigg[X_i^2\Bigg] - \Bigg\{\mathbb{E}\Bigg[X_i\Bigg]\Bigg\}^2 \\
&= \sigma^2\Delta_t - (\sigma\sqrt{\Delta_t}\,(2p-1))^2 \\
&= \sigma^2\Delta_t(1-(2p-1)^2)
\end{aligned} \tag{6}$$

The mean of the sum of two or more random variables is the sum of their means. The mean of the total change in the random walk using Equation (5) above is therefore...

$$\begin{aligned}
mean &= \sum_{i=1}^{N}\mathbb{E}\Bigg[X_i\Bigg] \\
&= N\sigma\sqrt{\Delta_t}\,(2p-1)
\end{aligned} \tag{7}$$

The variance of the sum of two or more independent random variables is the sum of their variances. The variance of the total change in the random walk using Equation (6) above and the fact that $\Delta_t = T/N$ is therefore...

$$\begin{aligned}
variance &= \sum_{i=1}^{N}\Bigg[\mathbb{E}\Bigg[X_i^2\Bigg] - \Bigg\{\mathbb{E}\Bigg[X_i\Bigg]\Bigg\}^2\Bigg] \\
&= N\sigma^2\Delta_t(1-(2p-1)^2) \\
&= \sigma^2 T(1-(2p-1)^2)
\end{aligned} \tag{8}$$

Using Equation (7) above the mean of the value of the scaled random walk at time T (i.e. the mean of M_T) is...

$$mean = M_0 + N\sigma\sqrt{\Delta_t}\,(2p-1) \tag{9}$$

Using Equation (8) above the variance of the value of the scaled random walk at time T (i.e. the variance of M_T) is...

$$variance = \sigma^2 T(1-(2p-1)^2) \tag{10}$$

The Scaled Symmetric Random Walk

To model the prices of risky assets we want to use a scaled symmetric random walk, which is a scaled random walk where the probability of an increase in the value of the random walk equals the probability of a decrease in the value of the random walk (i.e. $p = 0.50$).

Using Equation (9) above the mean of the value of the scaled symmetric random walk at time T is...

$$\begin{aligned}
mean &= M_0 + N\sigma\sqrt{\Delta_t}\,((2)(0.50)-1) \\
&= M_0
\end{aligned} \tag{11}$$

Using Equation (10) above the variance of the value of the scaled symmetric random walk at time T is...

$$\begin{aligned}
variance &= \sigma^2 T(1-((2)(0.50)-1)^2) \\
&= \sigma^2 T
\end{aligned} \tag{12}$$

The Probability Mass Function

If n is the number of trials, k is the number of successes and p is the probability of success then the equation for the probability mass function is...

$$P\Big[k\Big] = \frac{n!}{k!(n-k)!}\,p^k\,(1-p)^{n-k} \tag{13}$$

For example if the number of trials is 10 (n), the number successes is 7 (k) and the probability of each success is 0.60 (p) then the probability of 7 successes out of 10 trials is...

$$P\Big[7\Big] = \frac{10!}{7! \times 3!} \times 0.60^7 \times 0.40^3 = 0.215 \tag{14}$$

Equation (13) can be interpreted as....

Combinations that yield k successes		$\frac{n!}{k!(n-k)!}$
Probability of each individual success	x	$p^k\,(1-p)^{n-k}$
Cumulative probability	=	Prob[k]

A Hypothetical Problem

We have a scaled random walk with the following parameters...

M_0	=	Value of random walk at time zero	=	0.00
T	=	Number of time periods	=	2.00
N	=	Number of steps in the random walk	=	24.00
σ	=	Sigma value	=	0.20
p	=	Probability that random walk increases	=	0.70
Δ_t	=	Length of one time step (T/N)	=	0.08333

What is the mean and variance of the value of the scaled random walk at time T?

Using Equation (9) above the mean is...

$$\begin{aligned} mean &= 0 + N\sigma\sqrt{\Delta_t}\,(2p-1) \\ &= (24)(0.20)(0.2887)((2)(0.70)-1) \\ &= 0.5543 \end{aligned} \tag{15}$$

Using Equation (10) above the variance is...

$$\begin{aligned} variance &= \sigma^2 T(1-(2p-1)^2) \\ &= (.04)(2)(1-0.16) \\ &= 0.0672 \end{aligned} \tag{16}$$

What is the probability that the value of the random walk at time T will be greater than 1.00?

Up moves	Down moves	Value	Combinations	Indiv Prob	Cumul Prob
24	0	1.38	1	0.000192	0.000192
23	1	1.27	24	0.000082	0.001971
22	2	1.15	276	0.000035	0.009712
21	3	1.04	2024	0.000015	0.030523
Total					0.042398

Using Equation (13) above the probability that the value of the random walk at time T is greater than 1.00 is approximately 0.04.